Math Competition Study Plan

math competition Preparation

Student age:	12
Plan duration:	1 week
Daily study time:	1 hour

Contents

1. How to Use This Plan

This plan is designed for one focused week of math competition preparation. Here is how to get the most out of it.

1.1. Daily Structure

Each day follows a consistent rhythm:

Day	Focus
Day 1 – Day 4	One lesson introduction (theory + worked examples) followed by practice problems.
Day 5	Weekly review – no new concepts. Recap of all topics, mixed practice problems, and 2 challenge problems combining multiple topics.

1.2. How to Work Through a Lesson

1. **Read the theory section** fully before attempting any problems. Understand the concept, not just the formula.
2. **Study the worked examples** step by step. For each example, ask: "What was the key move here?" The **Key Insight** label names it explicitly.
3. **Attempt the practice problems yourself first** before reading the solution. Even a partial attempt builds understanding.
4. **Standard problems** build the core skill. Complete all of them.
5. **Challenge problems** (marked with a star ⋆) require an extra step or creative insight. Try them after the standard problems. It is fine to skip a challenge problem and return to it on Day 5.

1.3. When You Are Stuck

- Re-read the relevant theory section.
- Look at the most similar worked example and trace it step by step.
- If still stuck after 5 minutes on a standard problem, read the hint (if provided), then the solution.
- For challenge problems, being stuck is expected and normal. Write down what you **do** know, then look at the solution.

1.4. Problem-Solving Checklist

Use this checklist on any word problem or multi-step problem before you start calculating:

1. **What is the problem asking?** Identify the exact question.
2. **What do I know?** List the given numbers and facts.
3. **Is any information extra or distracting?** Not every number in a problem needs to be used.
4. **What should I find first?** Identify if there is a hidden intermediate value.
5. **What operation matches the *situation* (not just the keyword)?** "More" does not always mean add. Model the scenario.
6. **Does my answer make sense?** Check the magnitude and units of your result.

1.5. Progress Tracking

After each day, note which problems felt easy, which felt hard, and which felt impossible. This informs what topics to revisit and what to study next — the **What to Study Next** section gives you the natural continuation path.

2. Overview

2.1. Student Profile

2.1.1. Strengths

- strong with school math and arithmetic
- comfortable with fractions and decimals
- can solve routine word problems when the operation is clear

2.1.2. Focus Areas

- new to competition-style problems
- struggles when a problem has hidden conditions or extra information
- needs practice choosing a strategy before calculating
- not confident with counting cases, remainders, and multi-step logic

2.2. Why this plan?

This plan focuses on introducing competition-style problems and strategies, addressing the student's need to handle hidden conditions and multi-step logic. The week is sequenced to build foundational skills in counting and logic before tackling more complex topics like probability and modular arithmetic. By the end of the week, the student should be more confident in approaching competition problems with hidden conditions and multiple steps.

2.3. Milestones

- The student should be able to approach basic competition problems involving counting, logic, probability, and modular arithmetic with greater confidence.

2.4. What to Study Next

After completing this plan, continue with the following topics:

- Explore more advanced probability problems.
- Introduce geometry and its applications in competition problems.
- Develop deeper understanding of expressions and equations.
- Practice multi-step word problems with hidden conditions.
- Explore more complex patterns and sequences.

3. Schedule

Theme: Introduction to Competition Problem Solving

Topics: [counting], [logic-and-casework], [probability], [modular-arithmetic-introduction]

Goals:
- Develop skills in counting and logic for competition problems.
- Understand and apply basic probability concepts.
- Introduce modular arithmetic to handle remainders.

Day	Activities
Day 1	counting (30 min) · Counting Practice (30 min)
Day 2	logic-and-casework (30 min) · Logic Practice (30 min)
Day 3	probability (30 min) · Probability Practice (30 min)
Day 4	modular-arithmetic-introduction (30 min) · Modular Arithmetic Practice (30 min)
Day 5	Weekly Review (60 min)

4. Lessons

4.0.1. Day 1 — Systematic Counting: Choose the Structure Before Calculating

You are already strong with school math, arithmetic, fractions, and decimals, so today's goal is not harder calculation—it is better strategy. In math competition problems, counting often becomes tricky because the problem hides conditions like "not adjacent," "not allowed," or "repeats every 4." This lesson will help you slow down, choose a counting structure first, and avoid missing or double-counting cases.

4.0.1.1. Theory

Counting in competition math means finding all valid possibilities exactly once. Before calculating, ask: What am I counting? What choices are being made? Are there restrictions? A useful first move is to choose a structure: a list, a table, cases, a repeated cycle, or count-all-then-subtract.

For repeated patterns, identify the cycle length and decide whether the positions are 1-based or 0-based. In most contest problems, the first object is position 1. For example, if colors repeat Red, Blue, Green, Yellow, then the cycle length is 4. Position 23 has remainder 3 when 23 is divided by 4, so it matches the 3rd color, Green.

For choices, multiply only when the choices are independent and every combination is allowed. For example, if there are 3 shirts and 2 hats, then there are $3 \times 2 = 6$ shirt-hat outfits. If some combinations are not allowed, it is often easier to count all possible combinations first, then subtract the forbidden ones.

For hidden conditions, do not jump straight into arithmetic. Label the objects, make cases, and check whether each possibility is counted once. This is especially important for conditions like "next to," "before," "after," or "not allowed," because these conditions change the count.

4.0.1.2. Worked Examples

4.0.1.3. Example 1

Problem: A bead string repeats the colors Red, Blue, Green, Yellow in that order. The first bead is Red. What color is the 23rd bead?

Solution:

Step 1: The cycle is Red, Blue, Green, Yellow, so the cycle length is 4. Step 2: Divide 23 by 4. Since $23 = 5 \times 4 + 3$, the 23rd bead is in position 3 of the cycle. Step 3: The 3rd color is Green. Answer: Green.

Key insight: *Cycle indexing: find the remainder and match it to the correct position in the repeating cycle.*

4.0.1.4. Example 2

Problem: A snack pack contains one fruit, one drink, and one bar. There are 3 fruit choices, 2 drink choices, and 2 bar choices. However, packs with juice and a protein bar together are not allowed. How many snack packs are possible?

Solution:

Step 1: Count all possible packs without the restriction: $3 \times 2 \times 2 = 12$. Step 2: Count the forbidden packs. Juice is fixed and protein bar is fixed, but the fruit can be any of 3 choices, so there are 3 forbidden packs. Step 3: Subtract the forbidden packs: $12 - 3 = 9$. Answer: 9 snack packs.

Key insight: *Count all then subtract forbidden: handle the restriction after finding the easier total.*

4.0.1.5. Practice Problems

1. **A pattern repeats Circle, Square, Triangle in that order. The first shape is Circle. What is the 26th shape?**

   Solution: Step 1: The cycle length is 3. Step 2: Since $26 = 8 \times 3 + 2$, the 26th shape is in position 2 of the cycle. Step 3: Position 2 is Square. Answer: Square.

2. **A bead pattern repeats Red, Blue, Green, Yellow in that order. From position 1 through position 30, how many beads are Green?**

   Solution: Step 1: The cycle length is 4, and each full cycle has exactly 1 Green bead. Step 2: Since $30 = 7 \times 4 + 2$, there are 7 full cycles and 2 extra beads. Step 3: The 2 extra beads are Red and Blue, so they add no Green beads. Answer: 7 Green beads.

3. **A lunch combo has one sandwich, one fruit, and one drink. There are 3 sandwich choices, 2 fruit choices, and 3 drink choices. Combos with tuna and milk together are not allowed. How many lunch combos are possible? Explain your solution in 2–3 sentences. Use this format: I know _____. I need to find _____. First I _____ because _____. My answer makes sense because _____.**

   Solution: Step 1: Count all combos without the restriction: $3 \times 2 \times 3 = 18$. Step 2: Count the forbidden combos. Tuna is fixed and milk is fixed, but the fruit has 2 choices, so there are 2 forbidden combos. Step 3: Subtract: $18 - 2 = 16$. Answer: 16 lunch combos.

4. Challenge ⋆ — **Five boxes in a row are labeled 1 to 5. Three different stickers A, B, and C are placed in three of the boxes, at most one sticker per box. Stickers A and B may not be in adjacent boxes. How many arrangements are possible?** *Hint: First count all ways to place A, B, and C. Then subtract the arrangements where A and B are adjacent.*

   Solution: Step 1: Count all arrangements. Sticker A has 5 choices, sticker B has 4 remaining choices, and sticker C has 3 remaining choices, so there are $5 \times 4 \times 3 = 60$ arrangements. Step 2: Count the bad arrangements where A and B are adjacent. There are 4 adjacent pairs of boxes: 1–2, 2–3, 3–4, and 4–5. Step 3: For each adjacent pair, A and B can be ordered in 2 ways, and C can go in any of the 3 remaining boxes. So bad arrangements $= 4 \times 2 \times 3 = 24$. Step 4: Subtract: $60 - 24 = 36$. Answer: 36 arrangements.

5. Challenge ⋆ — **Seven seats in a row are numbered 1 to 7. Maya, Noah, and Priya sit in three of the seats. Priya cannot sit in seat 1 or seat 7. Maya must sit in a lower-numbered seat than Noah. Noah cannot sit in a seat numbered 1 more or 1 less than Priya's seat. How many seating arrangements are possible?** *Hint: Make cases based on Priya's seat. For each Priya seat, count possible Noah seats, then count how many lower-numbered seats Maya could use.*

Solution: Step 1: Priya can sit in seat 2, 3, 4, 5, or 6. Step 2: Count valid arrangements by Priya's seat. If Priya is in seat 2, Noah can be in 4, 5, 6, or 7. Maya has 2, 3, 4, and 5 choices respectively, for 14 total. If Priya is in seat 3, Noah can be in 1, 5, 6, or 7. Maya has 0, 3, 4, and 5 choices respectively, for 12 total. If Priya is in seat 4, Noah can be in 1, 2, 6, or 7. Maya has 0, 1, 4, and 5 choices respectively, for 10 total. If Priya is in seat 5, Noah can be in 1, 2, 3, or 7. Maya has 0, 1, 2, and 5 choices respectively, for 8 total. If Priya is in seat 6, Noah can be in 1, 2, 3, or 4. Maya has 0, 1, 2, and 3 choices respectively, for 6 total. Step 3: Add the cases: $14 + 12 + 10 + 8 + 6 = 50$. Answer: 50 seating arrangements.

4.0.1.6. Competition Tips

- Because your arithmetic is strong, your main competition move is to pause before calculating and name the structure: cycle, cases, table, or count-all-then-subtract.
- When a math competition problem says something like "not adjacent," "not allowed," or "must be before," treat it as a hidden condition. Count slowly and check that every arrangement is counted exactly once.
- For repeated patterns, always ask whether position 1 is the first item in the cycle. A remainder of 0 means the last item in the cycle, not the first.

4.0.2. Day 2 — Logic and Casework: Choosing Cases Before Calculating

Because you are strong with school math and arithmetic, the calculations in math competition problems usually will not be the hard part for you. Today's lesson focuses on the part you said is harder: noticing hidden conditions, avoiding extra-information traps, and choosing a strategy before calculating. Logic-and-casework is a core skill for competition problems because it helps you replace guessing with an organized list of possibilities.

4.0.2.1. Theory

Logic-and-casework means solving a problem by organizing the possible cases, then checking each case against the conditions. A case is one possible category or situation. Good cases must be complete, meaning every possible answer fits somewhere, and non-overlapping, meaning no answer gets counted twice.

A strong competition habit is: Classify before Analyze. First decide what your cases will be, then do the arithmetic inside each case. For example, to count the integers x with $3 < x \leq 8$, first list the possible integers: 4, 5, 6, 7, 8. If the problem also says x is even, check the list and keep 4, 6, 8, so there are 3 values.

Be careful with must versus could. If a condition says every A is B, then A must be inside B, but B does not have to be inside A. For example, if every student who solved Puzzle 1 also solved Puzzle 2, then Puzzle 1 solvers are included among Puzzle 2 solvers. It does not mean every Puzzle 2 solver solved Puzzle 1. Reversing an if-then statement is a common logic leak.

When a problem has several conditions, do not start by calculating randomly. Pick the condition that creates the fewest cases first. Then make a short table or list, cross off impossible cases, and check that your cases do not overlap.

4.0.2.2. Worked Examples

4.0.2.3. Example 1

Problem: How many integers x satisfy $2 < x \leq 9$ and are odd but not divisible by 3?

Solution:

Step 1: List the integers in the range $2 < x \leq 9$: 3, 4, 5, 6, 7, 8, 9. Step 2: Keep only the odd integers: 3, 5, 7, 9. Step 3: Remove the integers divisible by 3: 3 and 9 are removed. Step 4: The remaining values are 5 and 7, so there are 2 possible integers.

Key insight: *Filter cases: start with all possible values, then apply one condition at a time.*

4.0.2.4. Example 2

Problem: Two different integers a and b are chosen from 1 through 8. It is known that $a < b$, $a + b \leq 10$, and b is odd. How many ordered pairs (a, b) are possible?

Solution:

Step 1: Classify by the restrictive condition b is odd. Since $a < b$ and b is from 1 through 8, the possible odd values for b are 3, 5, and 7. Step 2: If $b = 3$, then a can be 1 or 2. Both give $a + b \leq 10$, so there are 2 pairs. Step 3: If $b = 5$, then a can be 1, 2, 3, or 4. All work, so there are 4 pairs. Step 4: If $b = 7$, then a must be less than 7 and $a + 7 \leq 10$, so $a \leq 3$. Thus a can be 1, 2, or 3, giving 3 pairs. Step 5: Total possible pairs: $2 + 4 + 3 = 9$.

Key insight: *Classify by the most restrictive condition first: choosing b first makes the cases short and non-overlapping.*

4.0.2.5. Practice Problems

1. **How many integers x satisfy $4 < x \le 10$ and are even?**

   Solution: Step 1: The integers with $4 < x \le 10$ are 5, 6, 7, 8, 9, 10. Step 2: The even ones are 6, 8, and 10. Step 3: There are 3 possible values.

2. **A one-digit integer n is greater than 2 and at most 8. It is not divisible by 2. How many possible values can n have?**

   Solution: Step 1: The one-digit integers greater than 2 and at most 8 are 3, 4, 5, 6, 7, 8. Step 2: Remove the values divisible by 2: 4, 6, and 8. Step 3: The remaining values are 3, 5, and 7, so there are 3 possible values.

3. **In a 9-student group, every student who solved Puzzle 1 also solved Puzzle 2. Exactly 4 students solved Puzzle 1, exactly 7 students solved Puzzle 2, and exactly 2 students solved neither puzzle. How many students solved Puzzle 2 but not Puzzle 1? Explain your solution in 2–3 sentences. Use this format: I know _____. I need to find _____. First I _____ because _____. My answer makes sense because _____.**

   Solution: Step 1: Since every Puzzle 1 solver also solved Puzzle 2, the 4 Puzzle 1 solvers are included in the 7 Puzzle 2 solvers. Step 2: The number who solved Puzzle 2 but not Puzzle 1 is $7 - 4 = 3$. Step 3: Check: 4 solved both, 3 solved only Puzzle 2, and 2 solved neither. The total is $4 + 3 + 2 = 9$, which matches the group size.

4. Challenge ⋆ — **How many two-digit numbers have digits from 1 through 9, have a tens digit less than the ones digit, and have digit sum at most 8?** *Hint: Classify by the tens digit. For each tens digit, list the possible ones digits that are larger and still keep the digit sum at most 8.*

   Solution: Step 1: Let the tens digit be a and the ones digit be b. We need $1 \le a < b \le 9$ and $a + b \le 8$. Step 2: If $a = 1$, then b can be 2, 3, 4, 5, 6, or 7: 6 choices. Step 3: If $a = 2$, then b can be 3, 4, 5, or 6: 4 choices. Step 4: If $a = 3$, then b can be 4 or 5: 2 choices. Step 5: If $a \ge 4$, then b must be greater than a, so $a + b$ is greater than 8. No more cases work. Step 6: Total: $6 + 4 + 2 = 12$.

5. Challenge ⋆ — **How many three-digit numbers can be made using digits from 1 through 6 with no digit repeated, if the hundreds digit is less than the tens digit and the ones digit is also less than the tens digit?** *Hint: The tens digit is the key case. Once you choose the tens digit, count how many smaller digits can go in the hundreds and ones places.*

   Solution: Step 1: Let the tens digit be b. The hundreds and ones digits must both be smaller than b, and they must be different from each other. Step 2: If $b = 1$ or $b = 2$, there are not two different smaller digits available, so there are 0 numbers. Step 3: If $b = 3$, the smaller digits are 1 and 2. They can be arranged in the hundreds and ones places in 2 ways. Step 4: If $b = 4$, there are 3 smaller digits. Choose an ordered pair for hundreds and ones: $3 \times 2 = 6$ ways. Step 5: If $b = 5$, there are 4 smaller digits, giving $4 \times 3 = 12$ ways. Step 6: If b =

6, there are 5 smaller digits, giving $5 \times 4 = 20$ ways. Step 7: Total: $0 + 0 + 2 + 6 + 12 + 20 = 40$.

4.0.2.6. Competition Tips

- In a math competition, do not start calculating until you can say what your cases are. Write a tiny case header such as x-values, tens digit, or solved Puzzle 1? before you analyze.
- Watch for must/could swaps. If the problem says every A is B, mark A inside B, but do not assume every B is A.
- When there is extra information, use it as a check after solving. If your case count does not match a total given in the problem, you may have overlapped cases or missed a case.

4.0.3. Day 3 — Probability by Counting Outcomes

You are already strong with school math, arithmetic, fractions, and decimals, so today's goal is not harder calculation; it is choosing what to count before calculating. In math competition problems, probability often hides a counting or casework problem, which connects directly to your current weakness with hidden conditions and multi-step logic. By the end of this lesson, you should be able to slow down, decide what the total outcomes are, count the favorable outcomes carefully, and then form the probability.

4.0.3.1. Theory

Probability measures how likely an event is. When all outcomes are equally likely, probability = number of favorable outcomes ÷ total number of outcomes. Write this as a fraction, such as 3/8 or 5/12, and simplify if possible.

The main competition move is: count first, calculate second. Do not start with a keyword. Instead, ask: What is one complete outcome? How many total outcomes are possible? Which outcomes satisfy every condition?

For example, if a spinner has numbers 1 through 10 and you want the probability of spinning an even number greater than 4, first count the total outcomes: 10. Then count the favorable outcomes: 6, 8, and 10, so there are 3 favorable outcomes. The probability is 3/10.

For multi-step situations, decide whether order matters. If a coin is flipped twice, HT and TH are different ordered outcomes. If two marbles are chosen at the same time, the pair red-blue is the same as blue-red, so unordered counting is usually better. Competition problems often become easier when you choose the right representation: list, table, cases, or complement. The complement means counting what you do not want, then subtracting from the total.

4.0.3.2. Worked Examples

4.0.3.3. Example 1

Problem: A fair spinner has 8 equal sections labeled 1 through 8. What is the probability that the spinner lands on a prime number or a multiple of 3?

Solution:

Step 1: Count the total outcomes. There are 8 equally likely sections. Step 2: List the prime numbers from 1 to 8: 2, 3, 5, 7. Step 3: List the multiples of 3 from 1 to 8: 3, 6. Step 4: Combine the lists without double-counting 3: 2, 3, 5, 6, 7. There are 5 favorable outcomes. Step 5: The probability is 5/8.

Key insight: *Avoid double-counting: when an outcome fits two descriptions, count it once.*

4.0.3.4. Example 2

Problem: A bag contains 3 red marbles, 2 blue marbles, and 1 green marble. Two marbles are drawn at the same time. What is the probability that the two marbles are different colors?

Solution:

Step 1: There are 6 marbles total. Since two are drawn at the same time, count unordered pairs. Step 2: Count all possible pairs: $6 \times 5 \div 2 = 15$. Step 3: It is easier to count the complement: pairs that are the same color. Step 4: Same-color red pairs: $3 \times 2 \div 2 = 3$. Same-color blue pairs: $2 \times 1 \div 2 = 1$. Same-color green pairs: 0. Step 5: Same-color pairs total $3 + 1 = 4$, so different-color pairs total $15 - 4 = 11$. Step 6: The probability is $11/15$.

Key insight: *Complement counting: count the unwanted same-color pairs, then subtract from the total.*

4.0.3.5. Practice Problems

1. **A box contains 5 yellow pencils and 3 purple pencils. One pencil is chosen at random. What is the probability that it is purple?**

   Solution: Step 1: Count the total pencils: $5 + 3 = 8$. Step 2: Count the favorable pencils: 3 purple pencils. Step 3: The probability is $3/8$.

2. **A fair spinner has 12 equal sections labeled 1 through 12. What is the probability that the spinner lands on an even number greater than 4?**

   Solution: Step 1: There are 12 total outcomes. Step 2: The even numbers greater than 4 are 6, 8, 10, and 12, so there are 4 favorable outcomes. Step 3: The probability is $4/12 = 1/3$.

3. **A card is chosen at random from cards numbered 1 through 20. What is the probability that the number is a multiple of 3 or a multiple of 5? Explain your solution in 2–3 sentences. Use this format: I know _____. I need to find _____. First I _____ because _____. My answer makes sense because _____.**

   Solution: Step 1: There are 20 total outcomes. Step 2: Multiples of 3 are 3, 6, 9, 12, 15, and 18, so there are 6. Step 3: Multiples of 5 are 5, 10, 15, and 20, so there are 4. Step 4: The number 15 was counted twice, so subtract 1. Step 5: Favorable outcomes: $6 + 4 - 1 = 9$. Step 6: The probability is $9/20$.

4. Challenge ⋆ — **A fair spinner has 5 equal sections labeled 1, 2, 3, 4, and 5. The spinner is spun twice. What is the probability that the second number is greater than the first number and the sum of the two numbers is even?** *Hint: A sum is even when the two numbers have the same parity: both odd or both even. Count ordered pairs because the first spin and second spin have different roles.*

   Solution: Step 1: Count total ordered outcomes. There are 5 choices for the first spin and 5 choices for the second spin, so $5 \times 5 = 25$ total outcomes. Step 2: The sum is even only when both numbers have the same parity. Step 3: Odd numbers are 1, 3, 5. Ordered pairs with the second greater are (1, 3), (1, 5), and (3, 5), giving 3 pairs. Step 4: Even numbers are 2, 4. Ordered pairs with the second greater are (2, 4), giving 1 pair. Step 5: Favorable outcomes total $3 + 1 = 4$. Step 6: The probability is $4/25$.

5. Challenge ⋆ — **Two fair six-sided dice are rolled. What is the probability that the product of the two numbers is divisible by 4, but the sum of the two numbers is not 7?** *Hint: First count the ordered dice rolls whose product is divisible by 4. Then remove the rolls among those whose sum is 7.*

   Solution: Step 1: There are $6 \times 6 = 36$ ordered outcomes. Step 2: Count rolls whose product is divisible by 4. A product fails to be divisible by 4 if both numbers are odd, or if one number is odd and the other is 2 or 6. Step 3: Both odd: $3 \times 3 = 9$ outcomes. One odd and one from $\{2, 6\}$: $3 \times 2 \times 2 = 12$ outcomes. So $9 + 12 = 21$ outcomes fail. Step 4: Therefore $36 - 21 = 15$ outcomes have product divisible by 4. Step 5: Now remove outcomes among those with sum 7. The sum-7 rolls are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1). Only (3,

4) and (4, 3) have product divisible by 4. Step 6: Favorable outcomes: $15 - 2 = 13$. Step 7: The probability is 13/36.

4.0.3.6. Competition Tips

- Because you are strong with arithmetic, do not rush into multiplying or adding. In a math competition, spend the first 10 seconds asking: What is one complete outcome, and does order matter?
- For hidden-condition probability problems, underline every condition and count outcomes that satisfy all of them. If two conditions are joined by "and," both must be true; if joined by "or," watch for overlap so you do not double-count.
- When the favorable outcomes are messy, try the complement. This is especially useful for your current weakness with multi-step logic because it turns many small cases into one cleaner subtraction.

4.0.4. Day 4 — Modular Arithmetic: Remainders, Cycles, and Hidden Patterns

You are already strong with school math and arithmetic, so today's goal is not harder calculation; it is choosing the right structure before calculating. Modular arithmetic is useful in math competition problems because many tricky questions hide a repeating cycle, a remainder, or a condition like "every 3rd position." Since you are new to competition-style problems with hidden conditions, this lesson will train you to ask: What is the cycle length? Which positions matter? Is the indexing 1-based or 0-based?

4.0.4.1. Theory

Modular arithmetic means working with remainders after division. For example, $38 = 3 \times 12 + 2$, so 38 has remainder 2 when divided by 3. In a repeating cycle of length 3, the 38th item lands in the same cycle position as remainder 2.

For a 1-based repeating pattern, use this rule: divide the position number by the cycle length. If the remainder is 1, choose the 1st item; if the remainder is 2, choose the 2nd item; and so on. If the remainder is 0, choose the last item in the cycle. Example: The pattern Red, Blue, Green repeats. The 14th item has $14 \div 3 = 4$ remainder 2, so the 14th item is Blue.

For circular boards or clocks, be careful with indexing. If spaces are numbered 1 through n, convert the start to a 0-based position by subtracting 1, add the number of steps, take the remainder when divided by n, then add 1 back. Example: On spaces 1 through 8, starting on space 6 and moving 11 spaces gives $((6 - 1 + 11)$ remainder $8) + 1 = (16$ remainder $8) + 1 = 0 + 1 =$ space 1.

Competition problems often ask for more than one position. Then do not list everything unless the list is short. Find the first position that fits, the last position that fits, and count by the cycle length. For example, numbers from 10 to 40 that have remainder 2 when divided by 6 are 14, 20, 26, 32, 38, so there are 5 such numbers.

4.0.4.2. Worked Examples

4.0.4.3. Example 1

Problem: A necklace pattern repeats Star, Circle, Triangle, Square. What is the 47th bead?

Solution:

Step 1: The cycle length is 4 because there are 4 bead types before the pattern repeats. Step 2: Divide 47 by 4: $47 = 4 \times 11 + 3$, so the remainder is 3. Step 3: In a 1-based cycle, remainder 3 means the 3rd item in the pattern. Answer: Triangle.

Key insight: *Cycle remainder: replace a faraway position with its remainder in one repeating block.*

4.0.4.4. Example 2

Problem: A banner repeats the colors Blue, Red, Green, Yellow, White. How many Yellow flags are in positions 20 through 83, inclusive?

Solution:

Step 1: The cycle length is 5, and Yellow is the 4th item in the cycle. Step 2: Yellow positions have remainder 4 when divided by 5: 4, 9, 14, 19, 24, … Step 3: The first Yellow position at least 20 is 24. The last Yellow position at most 83 is 79. Step 4: Count the sequence 24, 29, 34, …, 79. The number of jumps is $(79 - 24) \div 5 = 11$, so the number of terms is $11 + 1 = 12$. Answer: 12.

Key insight: *First-last-count: find the first matching position, the last matching position, then count by the cycle length.*

4.0.4.5. Practice Problems

1. **The actions Clap, Snap, Stomp repeat in that order. What is the 38th action?**

   Solution: Step 1: The cycle length is 3. Step 2: $38 = 3 \times 12 + 2$, so the remainder is 2. Step 3: Remainder 2 means the 2nd action in the cycle. Answer: Snap.

2. **A token starts on space 4 of a circular board numbered 1 through 9. Each move advances the token 5 spaces. What space is the token on after 17 moves?**

   Solution: Step 1: The token moves $17 \times 5 = 85$ spaces total. Step 2: Since the board has 9 spaces, $85 = 9 \times 9 + 4$, so moving 85 spaces is the same as moving 4 spaces. Step 3: Starting on space 4 and advancing 4 spaces lands on space 8. Answer: 8.

3. **A row of beads repeats Red, Blue, Green, Yellow. How many Green beads are in positions 10 through 37, inclusive? Explain your solution in 2–3 sentences. Use this format: I know _____. I need to find _____. First I _____ because _____. My answer makes sense because _____.**

   Solution: Step 1: The cycle length is 4, and Green is the 3rd item in the cycle. Step 2: Green positions have remainder 3 when divided by 4. Step 3: The Green positions from 10 through 37 are 11, 15, 19, 23, 27, 31, 35. Answer: 7.

4. Challenge ⋆ — **A code repeats A, B, C, D, E. Kai writes the first 86 symbols, so position 1 is A. Then he erases every symbol whose position number is divisible by 3. How many D symbols are left?** *Hint: First count all D positions. Then count only the D positions that also have position numbers divisible by 3.*

   Solution: Step 1: The cycle length is 5, and D is the 4th item, so D appears in positions 4, 9, 14, …, 84. Step 2: Count all D positions: $(84 - 4) \div 5 + 1 = 16 + 1 = 17$. Step 3: Now count erased D positions. The D positions divisible by 3 are 9, 24, 39, 54, 69, 84. Step 4: There are 6 erased D symbols, so $17 - 6 = 11$ D symbols are left. Answer: 11.

5. Challenge ⋆ — **Cards numbered 1 through 120 go through two repeating machines. The color machine repeats Red, Blue, Green, Yellow, so card 1 is Red. The letter machine repeats A, B, C, D, E, F, so card 1 is A. A card is special if it is Green and has letter E. What is the remainder when the sum of all special card numbers is divided by 10?** *Hint: Find the first card number that is both Green and E. Then notice how often both cycles line up again.*

   Solution: Step 1: Green is the 3rd item in a 4-cycle, so Green cards have positions 3, 7, 11, 15, … Step 2: E is the 5th item in a 6-cycle, so E cards have positions 5, 11, 17, 23, … Step 3: The first card that is both Green and E is 11. The color cycle and letter cycle line up again every 12 cards, so the special cards are 11, 23, 35, 47, 59, 71, 83, 95, 107, 119. Step 4: Pair the numbers: $11 + 119 = 130$, $23 + 107 = 130$, $35 + 95 = 130$, $47 + 83 = 130$, and 59

+ 71 = 130. The sum is 5 × 130 = 650. Step 5: 650 has remainder 0 when divided by 10. Answer: 0.

4.0.4.6. Competition Tips

- Because you are strong with arithmetic, pause before calculating: write the cycle length, the needed remainder, and whether the problem is using 1-based positions like 1st, 2nd, 3rd.
- In math competition problems with extra information, separate the conditions. For example, first find the positions for a symbol, then apply the extra condition such as “divisible by 3” or “between 10 and 37.”

4.0.5. Day 5 — Week 1 Review: Counting, Casework, Probability, and Remainders

This week you practiced the first tools that turn strong school math into competition math: organized counting, casework, probability, and remainders. Since you are already strong with arithmetic, today's goal is not faster calculation; it is choosing a strategy before calculating, especially when there are hidden conditions or extra information. By the end, you should be able to slow down, inventory the situation, and avoid missing or double-counting cases.

4.0.5.1. Theory

Week 1 cheat-sheet:

- Counting: Make a complete inventory of the sample space or choices. Keep order when order matters, such as 2 coins in order: HH, HT, TH, TT gives 4 outcomes.

- Logic and casework: Split the problem into clear cases. Count each case, check whether cases overlap, and combine only the valid cases. If two cases overlap, do not count the overlap twice.

- Probability using F.A.I.R.:
- Frame: Name the event, the experiment, and the answer form.
- Assess: Check whether outcomes are equally likely.
- Inventory: Count the full sample space.
- Ratio: P(event) = favorable / total. Also, P(not A) = 1 − P(A).

Standard inventories: 1 coin has 2 outcomes, 2 coins in order have 4 outcomes, 1 die has 6 outcomes, and 2 dice in order have 36 outcomes.

- Modular arithmetic and cycles: For a repeating cycle of length L, use the remainder after dividing by L to find the position in the cycle. For example, in the cycle Red, Blue, Green, Yellow of length 4, position 14 has remainder 2 when divided by 4, so it matches the 2nd color, Blue. If the remainder is 0, use the last item in the cycle.

- Competition habit: Do not jump to calculation. First ask, "What am I counting?" and "Are there overlapping cases?"

4.0.5.2. Worked Examples

4.0.5.3. Example 1

Problem: A fair coin is flipped twice in order, and then a fair die is rolled. What is the probability that the coin flips show exactly one head and the die shows an even number?

Solution:

Step 1: Inventory the coin outcomes. The ordered coin outcomes are HH, HT, TH, TT, so there are 4 coin outcomes. Step 2: Exactly one head occurs in HT and TH, so there are 2 favorable coin outcomes. Step 3: A die has 6 outcomes. The even outcomes are 2, 4, and 6, so there are 3 favorable die outcomes. Step 4: The full experiment has $4 \times 6 = 24$ equally likely outcomes. Favorable outcomes: $2 \times 3 = 6$. Step 5: Probability $= 6/24 = 1/4$.

Key insight: *Inventory before ratio: count the full ordered sample space first, then count favorable outcomes.*

4.0.5.4. Example 2

Problem: Positions 1 through 30 are colored in the repeating cycle Red, Blue, Green, Yellow. How many positions are either Blue or multiples of 3?

Solution:

Step 1: The cycle length is 4. Blue is the 2nd color, so Blue positions have remainder 2 when divided by 4. Step 2: Blue positions from 1 to 30 are 2, 6, 10, 14, 18, 22, 26, 30. There are 8. Step 3: Multiples of 3 from 1 to 30 are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. There are 10. Step 4: Some positions were counted in both lists: 6, 18, and 30. There are 3 overlaps. Step 5: Total either Blue or multiple of 3 $= 8 + 10 - 3 = 15$.

Key insight: *Overlap check: when counting "either A or B," subtract positions that were counted twice.*

4.0.5.5. Practice Problems

1. **A code is made by choosing one letter from A, B, C and then one digit from 1, 2. How many different codes are possible?**

   Solution: Step 1: List by letter. A gives A1, A2; B gives B1, B2; C gives C1, C2. Step 2: There are $2 + 2 + 2 = 6$ codes. Answer: 6.

2. **A fair die is rolled once. What is the probability of not rolling a number less than 3?**

   Solution: Step 1: The numbers less than 3 are 1 and 2, so there are 2 outcomes to avoid. Step 2: The outcomes that are not less than 3 are 3, 4, 5, and 6, so there are 4 favorable outcomes. Step 3: Probability $= 4/6 = 2/3$. Answer: $2/3$.

3. **Positions 1 through 25 are labeled with the repeating cycle A, B, C. How many positions are either labeled C or are multiples of 4? Explain your solution in 2–3 sentences. Use this format: I know _____. I need to find _____. First I _____ because _____. My answer makes sense because _____.**

   Solution: Step 1: The cycle length is 3, and C is the 3rd label, so C appears at positions 3, 6, 9, 12, 15, 18, 21, 24. There are 8. Step 2: Multiples of 4 from 1 to 25 are 4, 8, 12, 16, 20, 24. There are 6. Step 3: Positions 12 and 24 were counted in both groups, so subtract 2 overlaps. Step 4: Total $= 8 + 6 - 2 = 12$. Answer: 12.

4. Challenge ⋆ — **A number is chosen at random from the integers 1 through 24. What is the probability that the number is a multiple of 2 or leaves remainder 1 when divided by 4, but is not a multiple of 3?** *Hint: First count the numbers that satisfy the "multiple of 2 or remainder 1" condition. Then remove the numbers among them that are multiples of 3.*

   Solution: Step 1: Multiples of 2 from 1 to 24: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24. There are 12. Step 2: Numbers that leave remainder 1 when divided by 4 are 1, 5, 9, 13, 17, 21. There are 6. Step 3: These two groups do not overlap because a number with remainder 1 when divided by 4 is odd, while multiples of 2 are even. So there are $12 + 6 = 18$ numbers before the "not multiple of 3" condition. Step 4: Remove multiples of 3 from those groups. Even multiples of 3 are 6, 12, 18, 24, giving 4 numbers. Remainder-1 mod 4 multiples of 3 are 9 and 21, giving 2 numbers. Step 5: Favorable numbers $= 18 - 6 = 12$. Probability $= 12/24 = 1/2$. Answer: $1/2$.

5. Challenge ⋆ — **Maya writes the integers 1 through 40. She circles every number that is even or leaves remainder 1 when divided by 5. Then she randomly chooses one of the circled numbers. What is the probability that the chosen number is divisible by 3?** *Hint: The denominator is not 40. First count how many numbers are circled, then count how many circled numbers are divisible by 3.*

   Solution: Step 1: Count the circled numbers. There are 20 even numbers from 1 to 40. Step 2: Numbers that leave remainder 1 when divided by 5 are 1, 6, 11, 16, 21, 26, 31, 36. There are 8. Step 3: Some of these are also even: 6, 16, 26, 36. There are 4 overlaps. So total circled numbers $= 20 + 8 - 4 = 24$. Step 4: Count circled numbers divisible by 3. Even multiples of 3 from 1 to 40 are 6, 12, 18, 24, 30, 36. There are 6. Step 5: Numbers divisible by 3 and leaving remainder 1 when divided by 5 are 6, 21, 36. There are 3. Step 6: The overlap of those two favorable groups is 6 and 36, so subtract 2. Favorable circled numbers divisible by $3 = 6 + 3 - 2 = 7$. Step 7: Probability $= 7/24$. Answer: $7/24$.

4.0.5.6. Competition Tips

- In your math competition, pause for 5 seconds before calculating and name the experiment or set you are counting. This helps prevent the common mistake of using the wrong denominator in probability.
- When a problem says “or,” make an overlap check. Count case A, count case B, then ask whether anything was counted twice.
- For remainder or cycle problems, write a few early terms or positions first. Because you are strong with arithmetic, your main risk is not computation; it is choosing the wrong structure too quickly.